RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

M.A./M.SC. FIRST SEMESTER EXAMINATION, JANUARY 2013

FIRST YEAR

MATHEMATICS

Date : 11/01/2013

Time : 11.00 am – 1.00 pm

Paper : II

Full Marks : 50

All questions carry equal marks

Answer **any ten** questions

1. Let V be a vector space. Define a flat A in V and prove that A is a flat if and only if it is closed under the affine combination of vectors in V. For a vector subspace W of V and its cosets $v_i + W$, for $i = 1, 2$, prove that, either $(v_1 + W) \cap (v_2 + W) = \emptyset$ or $(v_1 + W) = (v_2 + W)$.

2. Let T be a linear operator on $P_2(\mathbb{R})$ defined by
$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2.$$
Beginning with the standard ordered basis α of $P_2(\mathbb{R})$ find an ordered basis β for which
$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

3. Consider the vector space $P_2(\mathbb{R})$. Let $B_1 = \{g_1, g_2, g_3\}$ and $B_2 = \{h_1, h_2, h_3\}$, where $g_1 = 1, g_2 = -1 + x, g_3 = 1 - x + x^2$, and $h_1 = 1 - x - x^2, h_2 = 3x - 2x^2, h_3 = 1 - 2x^2$. Given that $[p(x)]_{B_1} = (1, 4, 3)^T$, find $[p(x)]_{B_2}$.

4. Consider the set of all linear operators $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that T preserves the line $y = x$. Find the dimension of this subspace of $M_{2\times2}(\mathbb{R}^2)$. How many 2×2 invertible matrices are there with entries from a finite field of three elements?

5. Define the dual space V^* of a vector space V with basis $\alpha = \{v_1, v_2, \ldots, v_n\}$. Defining the co-ordinate functions v_i^* with respect to the basis α, prove that the set $\alpha^* = \{v_1^*, v_2^*, \ldots, v_n^*\}$ of co-ordinate functions forms a basis for the dual space V^*, where for any $T \in V^*$, one has $T = \sum_{i=1}^n T(v_i)\, v_i^*$.

6. Define Hermitian inner product over a complex vector space V. For $\alpha, \beta \in V$ prove that $\langle\alpha, \beta\rangle = \Re\langle\alpha, \beta\rangle + i\,\Re\langle\alpha, i\beta\rangle$, (where $\Re z$ stands for the real part of the complex number z) and hence prove the polarization identity $\langle\alpha, \beta\rangle = \frac{1}{4}\sum_{n=1}^4 i^n||\alpha + i^n\beta||^2$.

7. Consider the inner product space H of all continuous complex valued functions defined on $[0, 2\pi]$ under the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}\, dt.$$

Show that $S = \{e^{int} : n \text{ is an integer}\}$ is an orthonormal set in H and compute the Fourier coefficients of $f(t) = t$ relative to S.

8. State and prove the Bessel's inequality in an inner product space. When does it become an equality?

9. For two subspaces U and W of a vector space V, define the *projection* of V onto U along W. Prove that a linear transformation $T : V \to V$ is a projection if and only if $T = T^2$. Let $V = P_3(\mathbb{R})$ with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$ for all $f(x), g(x) \in V$. Compute the orthogonal projection $f_1(x)$ of $f(x) = x^3$ on $V = P_2(\mathbb{R})$. [Given that $\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2 - 1)\}$ is an orthonormal basis for $V = P_2(\mathbb{R})$.]

10. Let $f : V \to V$ be a rigid motion on a finite dimensional real inner product space V. Then prove that there exists a unique orthogonal operator T on V and a unique translation g on V such that $f = g \circ T$.

11. Let V be a finite dimensional inner product space over F and let g be a linear functional on V. Prove that there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$. Also prove that $y \in N(g)^{\perp}$, the orthogonal complement of the null space of g.

12. Prove that the mapping $T \mapsto T^*$ is a conjugate-linear anti-isomorphism of period 2, where T^* is the adjoint of the linear operator T on a finite dimensional inner product space V.

13. Let V be a finite dimensional inner product space and E be an orthogonal projection of V on a subspace W. Prove that E is self-adjoint. If T is a linear operator on V, prove that if T has an eigen vector then so does T^*.

14. For two Hermitian matrices H_1 and H_2, show that $A = H_1 + iH_2$ is normal if and only if $H_1H_2 = H_2H_1$. Prove that if an upper triangular matrix is normal, then it must be a diagonal matrix.

15. Find the spectral decomposition of the normal matrix $A = \begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{bmatrix}$. [Given that the characteristic polynomial of A is $(\lambda - i)^2(\lambda + i)$.]